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Elastic spherical shell impacted with an elastic barrier: A closed form solution

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ABSTRACT

In this paper, the impact of a thin-walled elastic spherical shell with an elastic barrier is investigated. Hertzian and Reissner (membrane–bending deformation) theories were employed for the deformation equations. Due to the complexity of the equations a linearization of the equations was proposed and a closed form solution of the problem was obtained. Newtonian method is applied in order to obtain the impact force and the time duration. The closed-form solution enables one to parametrically study the impact and the related quantities. Finally the results from the analytical solution are validated by the finite element method and also are compared with the results presented in Young (2003). The comparison of the results reveals a good agreement. It is concluded that the proposed closed-form solution can be used to parametrically assess the impact of elastic spherical shells to elastic half space.

1. Introduction

Contact problems of two deformable elastic bodies have been studied for decades. These problems become more complex when the bodies are impacted to one another, due to the transient and nonlinear nature of the problem. In particular, impact analysis of spherical shells is important and has attracted investigator's attention due to their wide range of applications in solid mechanics and biomechanics.

The impact loading of spherical shells has been the subject of many theoretical and experimental studies in engineering. Many attempts have been made to employ spherical shell theory and to perform impact analyses. For example, Reissner (1947) obtained explicit results for shells with and without edge restraints carrying either point or distributed loads. He concluded that center deflection is a function of shell radius 'h' to shell thickness 't'. Furthermore Reissner (1959) investigated the solution to some problems using membrane theory and by solving the equation of flexure for cantilever spherical shell. Engin (1969) proposed an analytical model which included bending and membrane compliance of the shell and evaluated the response for a fluid filled shell subjected to delta-function type loading. In addition the cases of a hollow spherical shell were also considered by Engin. In the experimental studies of Kenner and Goldsmith (1972), they obtained the magnitude of the impact force and the strains at different locations for both fluid filled and hollow spherical shells. Kunukkasseril and Palaninathan (1975) conducted impact experiments on shallow spherical shells to produce different pulse durations, they measured the impact force and strains. Hammel (1976) developed and solved a linear integral equation for an unknown impact force. He considered a particle striking a spherical shell, and concluded that the elastic deformation of the shell was much less than the deformations of a plate of equal thickness due to the same impact load. Senitskii (1982) developed Hammel's approach further by including the effect of local deformations in the shell due to Hertzian stress field. Stein and Wriggers (1982) studied impact-contact problems of thin elastic shells taking into account geometrical nonlinearities within the contact region.

Numerical results were also obtained for the impact-contact problem of spherical shells. Koller and Busenhart (1986) investigated elastic impact of spheres on thin spherical shells. A nonlinear integral–differential equation of the impact process was developed on the basis of Reissner's approximate theory and the quasi-static Hertzian contact theory. Dynamic buckling of a thin shallow spherical shell under impact loads was numerically calculated by Chun et al. (1992). They concluded that the critical buckling load increased with the enlargement of the loading area. Sabodash and Zhemkova (1993) investigated the dynamic reaction of a spherical shell which was subjected to the local effect of a normal pressure pulse. A numerical method based on the characteristic relationships was developed. Trial calculations were performed, and the numerical results were analyzed. Consequently a number of mechanical effects of practical and scientific interest were established by Sabodash and Zhemkova (1993). Pauchard and Rica (1998) studied the deformation of a thin elastic shell striking a rigid plane, and in another case, he investigated the elastic shell subjected to a localized load based on the total elastic energy. He also determined the restitution coefficient of the shell during impact, in his research

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coefficient of restitution was determined other than one due to the dissipative energy caused by the friction between the shell and the rigid plane. Young (2003) performed a study regarding a fluid filled spherical shell of arbitrary thickness impacting to an elastic sphere. The model was based on combining the Hertzian contact stiffness and the effective local membrane and bending stiffness to derive implicit formulations for global impact characteristics.

This study emanated from the biomechanics of human head subjected to a blunt impact and in particular the traumatic brain injuries as a result of the impact. In the present study, the analysis of a thin walled elastic spherical shell when it is impacted with an elastic barrier is presented, and a closed form solution for this problem is proposed. Closed form solutions are important for parametric studies of problems, i.e., the influence of each parameter of the problem on the results can be evaluated. The closed form solution of this study was validated with the finite element method and the results were compared with Young's (2003). It is important to note that young's paper formulated an implicit equation for transmitted force during the maximum compression of an elastic shell based on conservation of mechanical energy; thus it is necessary to employ a numerical method in order to solve the equation and to obtain the maximum transmitted force. Therefore, a parametric study for this problem (the implicit equation) is not feasible. However, the method and formulation of the problem presented in this paper is based on kinematics consideration and Newtonian method and provides an explicit expression for important characteristics of an elastic spherical shell subjected to an impact. Consequently, there is no need to employ a numerical method to achieve impact parameters. That is, the closed form solution presented in this paper facilitates parametric study of problems involving the impact of a thin-walled elastic spherical shell with an elastic barrier. Note that closed form solutions of engineering problems are essential for investigators.

To the best of authors' knowledge, a closed form solution of the impact of a thin-walled spherical shell with an elastic barrier based on this method has not been addressed in the literature, i.e., developing expressions to predict impact parameters of elastic spherical shell such as: transmitted force, time duration and elastic deformation. In the following section a closed form response of the impact of a thin walled elastic spherical shell with an elastic barrier is presented. In Section 3 the results are presented, followed by the validation of the analytical approach in Section 4. The conclusions and remarks are presented in Section 5.

2. Analysis

Consider a thin-walled elastic spherical (hollow) shell having an outer radius of R_{sh}, and a thickness of h, which is traveling at a constant velocity V_0 towards an elastic flat barrier under frictionless conditions. As the shell comes in contact with the elastic barrier, the spherical shell deforms due to its elasticity, and in the rebound, it returns to its initial form. Fig. 1 illustrates the bodies.

In this problem the deflection of the shell versus time is symmetric with respect to the axis of contact (the line connecting the center of the shell and the point of contact). It is assumed that no plastic deformation occurs during the contact. Consequently, during the impact of the elastic spherical shell, two events take place: approach (elastic deformation) of the shell due to Hertzian contact deflection of elastic wall and shell, and Reissner membrane–bending deflection of the shell. Therefore the total deformation, i.e., the displacement of the center of the shell, is the sum of both the local Hertzian deformation and Reissner compliance of contact area. Thus,

$$\Delta x = \alpha + \beta, \tag{1}$$

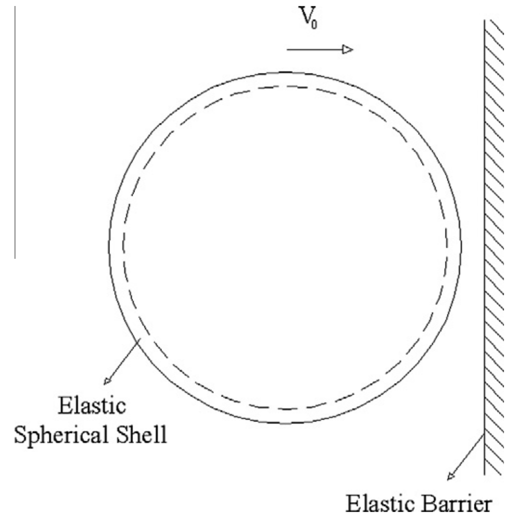

Fig. 1. Thin elastic spherical shell moves towards an elastic half space.

where Δx is the total deflection or center displacement of the shell during the contact, α is the mutual approach of elastic shell and half space due to the Hertzian contact stiffness and, β is the elastic deformation of the shell due to Reissner's effect, i.e., membrane–bending deformation.

According to the Hertzian contact theory, there is a relationship between the transmitted force and the approach of two elastic bodies during the contact (Johnson, 1972), therefore, the transmitted force F is,

$$F = K_2\alpha^{\frac{3}{2}}, \tag{2}$$

where

$$K_2 = \frac{4}{3\pi}\cdot\frac{\sqrt{R_{sh}}}{(\delta_{sh}+\delta_{sol})} \tag{3}$$

and where

$$\delta_{sh} = \frac{1-\nu_{sh}^2}{\pi E_{sh}} \tag{4}$$

and

$$\delta_{sol} = \frac{1-\nu_{sol}^2}{\pi E_{sol}} \tag{5}$$

and where ν_{sh}, ν_{sol} are Poisson's ratios of the spherical shell and elastic barrier, respectively, and E_{sh}, E_{sol} are Elasticity modulus of the spherical shell and barrier (Half space), respectively.

The above expressions (Eqs. (2)–(5)) are applicable for continuous smooth and frictionless bodies providing:

- The ratio of the maximum radius of contact area, r, to the outer radius of spherical shell is quite small $\frac{r}{R_{sh}} \ll 1$.
- Elastic deformation must remain small in comparison with the geometry of each colliding object, therefore $\frac{\alpha}{R_{sh}} \ll 1$.

On the other hand for a thin hollow spherical shell, membrane and bending deflection resulted from a force F applied as a uniform pressure on a small area (with radius r) has the linear equation for small deflections (Reissner, 1947):

$$F = K_{sh}\beta, \tag{6}$$

where

$$K_{sh} = \frac{2.3E_{sh}h^2}{R_{sh}\sqrt{1-\nu_{sh}^2}}. \tag{7}$$

In fact Eq. (6) is a linear relationship between the transmitted force and the membrane and bending deformation. Reissner theory for thin shell is an excellent estimation for the following cases:

– The thickness ratio (h/R_{sh}) smaller than 0.1,
– the ratio of the radius of loading area (r) to the outer radius is quite small $\frac{r}{R_{sh}} \ll 1$, and
– membrane and bending deformation are small compared with outer radius of spherical shell, then: $\frac{\beta}{R_{sh}} \ll 1$.

These conditions are suitable for low speed impact of elastic shell during contact. Therefore, this problem is analyzed assuming low velocity impact and quasi-static conditions.

When the combination of Hertz and Reissner theories is applied to the shell, the total deflection should be much smaller than other dimension, therefore $\Delta x/R_{sh} \ll 1$.

Taking the time derivative of Eq. (1), leading to the velocity of the center of mass of the shell V'_{sh} as,

$$\frac{d}{dt}(\Delta x) = V'_{sh} = \frac{d\alpha}{dt} + \frac{d\beta}{dt}. \tag{8}$$

Taking the time derivation of Eq. (8) and applying Newtonian Law and Hertz theory leads to,

$$\frac{dV'_{sh}}{dt} = \frac{d^2\alpha}{dt^2} + \frac{d^2\beta}{dt^2} = \frac{-F}{m_{sh}} = -K_1K_2\alpha^{\frac{3}{2}}, \tag{9}$$

where m_{sh} is the mass of the spherical shell, and $\frac{dV'_{sh}}{dt}$: uniform acceleration of the shell due to impact, $\frac{d^2\alpha}{dt^2}$, $\frac{d^2\beta}{dt^2}$: accelerations of the approach and the membrane and bending deformation, respectively, and,

$$K_1 = \frac{1}{m_{sh}}. \tag{10}$$

On the other hand, from Eqs. (2) and (6) the membrane and bending deformation is,

$$\beta = \frac{K_2}{K_{sh}}\alpha^{\frac{3}{2}}. \tag{11}$$

Substitution of β from Eq. (11) into Eq. (9) leads to a highly nonlinear and complex differential equation. The resulted nonlinear equation cannot be analytically solved and a closed form solution cannot be achieved. To overcome this difficulty, Eq. (2) is linearized by introducing a coefficient K_L as,

$$F = K_L\alpha, \tag{12}$$

where K_L is an unknown and will be determined using the equivalent elastic potential energy method. Elastic potential energies for both linear and non-linear cases are calculated up to the maximum approach.

Let $\alpha_{\max}$ be the maximum elastic deformation (the maximum approach), then, the elastic potential energies for the nonlinear and linear cases are:

$$U_{Nonlinear} = \int_0^{\alpha_{\max}} K_2\alpha^{\frac{3}{2}}d\alpha = \frac{2}{5}K_2\alpha_{\max}^{\frac{5}{2}} \tag{13}$$

and,

$$U_{Linear} = \int_0^{\alpha_{\max}} K_L\alpha d\alpha = \frac{1}{2}K_L\alpha_{\max}^2. \tag{14}$$

However, the maximum approach between two elastic solids is determined by Johnson (1972) as,

$$\alpha_{\max} = \left(\frac{5}{4}\frac{V_0^2}{K_1K_2}\right)^{\frac{2}{5}}. \tag{15}$$

Equating the elastic potential energies of Eqs. (13) and (14) and using Eq. (15) leads to the linearized K_L as,

$$K_L = \frac{V_0^2}{K_1}\left(\frac{4}{5}\frac{K_1K_2}{V_0^2}\right)^{\frac{4}{5}}. \tag{16}$$

Consequently, instead of nonlinear Eq. (11), Eq. (17) is employed.

$$\beta = \frac{K_L}{K_{sh}}\alpha. \tag{17}$$

Table 1
Comparison of the results with different methods, for $\delta = 0.05$; $E_{sol} = E_{sh} = 200$ GPa; $V_{Upper} = 1.67$ m/s.

V_0 (m/s)	$\Delta x_{\max}$ (Young, 2003) (mm)	$U = \Delta x_{\max}$ closed form solution (mm)	$U_T = \Delta x_{\max}$ FEM (mm)	$\lvert U_T - U\rvert$	Percentage of difference between U and U_T (%)	t_f closed from solution (ms)	t_f FEM (ms)	Percentage of difference t_f's (%)	$F_{\max}$ closed form solution (KN)	$F_{\max}$ (Young, 2003) (KN)	Percentage of difference $F_{\max}$'s (%)
0.2	0.0257	0.0278	0.0251	0.0027	9.7	0.4	0.371	7.2	1.06	0.97	8.4
0.4	0.0491	0.0527	0.0497	0.003	5.6	0.391	0.369	5.6	2.18	2.02	7.3
0.6	0.0716	0.0765	0.0698	0.0067	8.7	0.384	0.363	5.4	3.34	3.1	7.1
0.8	0.0944	0.1	0.0919	0.0081	8.1	0.383	0.3625	5.3	4.46	4.16	6.7
1	0.116	0.123	0.121	0.002	1.6	0.382	0.362	5.2	5.6	5.25	6.2
1.2	0.138	0.146	0.146	0	0	0.381	0.3617	5	6.74	6.34	5.9
1.4	0.161	0.169	0.161	0.008	4.7	0.38	0.361	5	7.88	7.44	5.5

Table 2
Comparison of the results with different methods, for $\delta = 0.05$; $E_{sol} \to \infty$ $E_{sh} = 200$ GPa; $V_{Upper} = 1.83$ m/s.

V_0 (m/s)	$\Delta x_{\max}$ (Young, 2003) (mm)	$U = \Delta x_{\max}$ closed form solution (mm)	$U_T = \Delta x_{\max}$ FEM (mm)	$\lvert U_T - U\rvert$	Percentage of difference between U and U_T (%)	t_f Closed from solution (ms)	t_f FEM (ms)	Percentage of difference t_f's (%)	$F_{\max}$ closed form solution (KN)	$F_{\max}$ (Young, 2003) (KN)	Percentage of difference $F_{\max}$'s (%)
0.2	0.0236	0.0251	0.022	0.0031	12.3	0.383	0.352	8	1.11	1.04	6.3
0.4	0.045	0.048	0.048	0	0	0.379	0.3515	7.2	2.25	2.13	5.3
0.6	0.067	0.0701	0.067	0.0031	4.4	0.3788	0.3507	7.4	3.39	3.24	4.4
0.8	0.088	0.091	0.089	0.002	2.1	0.3787	0.35	7.5	4.52	4.36	3.5
1	0.11	0.113	0.121	0.008	6.6	0.378	0.348	7.9	5.64	5.48	2.8
1.2	0.131	0.134	0.145	0.011	7.5	0.376	0.346	7.9	6.77	6.6	2.5
1.4	0.152	0.153	0.157	0.004	2.5	0.373	0.344	7.7	7.89	7.73	2

Table 3
Comparison of the results with different methods, for $\delta = 0.05$; $E_{sol} = 200 \times 10^9$ $E_{sh} = 70$ GPa; $V_{Upper} = 1.77$ m/s.

V_0 (m/s)	Δx_{max} (Young, 2003) (mm)	$U = \Delta x_{max}$ closed form solution (mm)	$U_T = \Delta x_{max}$ FEM (mm)	$\lvert U_T - U\rvert$	Percentage of difference between U and U_T (%)	t_f closed from solution (ms)	t_f FEM (ms)	Percentage of difference t_f's (%)	F_{max} closed form solution (KN)	F_{max} (Young, 2003) (KN)	Percentage of difference F_{max}'s (%)
0.2	0.024	0.025	0.024	0.001	4	0.385	0.3575	7.1	0.38	0.35	7.8
0.4	0.0465	0.049	0.047	0.002	4.1	0.378	0.357	5.5	0.78	0.73	6.4
0.6	0.068	0.072	0.070	0.002	2.8	0.376	0.355	5.5	1.18	1.11	5.9
0.8	0.089	0.094	0.091	0.003	3.2	0.375	0.351	6.4	1.57	1.49	5
1	0.111	0.116	0.109	0.007	6	0.3749	0.344	8.2	1.97	1.88	4.5
1.2	0.132	0.137	0.134	0.003	2.2	0.3747	0.343	8.7	2.37	2.27	4.2
1.4	0.153	0.159	0.16	0.001	0.6	0.3746	0.342	8.9	2.76	2.66	3.6

Table 4
Comparison of the results with different methods, for $\delta = 0.08$; $E_{sol} = E_{sh} = 200$ GPa; $V_{Upper} = 1.86$ m/s.

V_0 (m/s)	Δx_{max} (Young, 2003) (mm)	$U = \Delta x_{max}$ closed form solution (mm)	$U_T = \Delta x_{max}$ FEM (mm)	$\lvert U_T - U\rvert$	Percentage of difference between U and U_T (%)	t_f closed from solution (ms)	t_f FEM (ms)	Percentage of difference t_f's (%)	F_{max} closed form solution (KN)	F_{max} (Young, 2003) (KN)	Percentage of difference F_{max}'s (%)
0.2	0.024	0.026	0.023	0.003	11.5	0.367	0.361	1.6	1.81	1.66	8.2
0.4	0.045	0.049	0.045	0.004	8.2	0.344	0.357	3.6	3.85	3.52	8.5
0.6	0.065	0.07	0.068	0.002	2.8	0.333	0.356	6.4	5.97	5.45	8.7
0.8	0.085	0.092	0.091	0.001	1.1	0.327	0.355	7.8	8.12	7.42	8.6
1	0.104	0.112	0.114	0.002	1.7	0.322	0.352	8.5	10.29	9.42	8.4
1.2	0.123	0.133	0.137	0.004	2.9	0.319	0.346	7.8	12.49	11.43	8.4
1.4	0.142	0.153	0.159	0.006	3.7	0.316	0.341	7.3	14.69	13.46	8.3

Table 5
Comparison of the results with different methods, for $\delta = 0.08$; $E_{sol} \to \infty$ $E_{sh} = 200$ GPa; $V_{Upper} = 2.07$ m/s.

V_0 (m/s)	Δx_{max} (Young, 2003) (mm)	$U = \Delta x_{max}$ closed form solution (mm)	$U_T = \Delta x_{max}$ FEM (mm)	$\lvert U_T - U\rvert$	Percentage of difference between U and U_T (%)	t_f closed from solution (ms)	t_f FEM (ms)	Percentage of difference t_f's (%)	F_{max} closed form solution (KN)	F_{max} (Young, 2003) (KN)	Percentage of difference F_{max}'s (%)
0.2	0.021	0.023	0.021	0.002	8.7	0.327	0.337	2.9	2.03	1.85	8.8
0.4	0.04	0.043	0.043	0	0	0.314	0.3368	6.7	4.22	3.87	8.2
0.6	0.058	0.063	0.064	0.001	1.5	0.308	0.3364	8.4	6.46	5.94	8
0.8	0.076	0.082	0.086	0.004	4.6	0.305	0.336	9.2	8.71	8.04	7.6
1	0.094	0.101	0.107	0.006	5.6	0.302	0.335	9.8	10.97	10.16	7.3
1.2	0.112	0.12	0.129	0.009	6.9	0.301	0.332	9.3	13.23	12.29	7.1
1.4	0.13	0.138	0.15	0.012	8	0.299	0.3305	9.5	15.5	14.43	6.9

Table 6
Comparison of the results with different methods, for $\delta = 0.08$; $E_{sol} = 200 \times 10^9$ GPa $E_{sh} = 70$ GPa; $V_{Upper} = 1.99$ m/s.

V_0 (m/s)	Δx_{max} (Young, 2003) (mm)	$U = \Delta x_{max}$ closed form solution (mm)	$U_T = \Delta x_{max}$ FEM (mm)	$\lvert U_T - U\rvert$	Percentage of difference between U and U_T (%)	t_f closed from solution (ms)	t_f FEM (ms)	Percentage of difference t_f's (%)	F_{max} closed form solution (KN)	F_{max} (Young, 2003) (KN)	Percentage of difference F_{max}'s (%)
0.2	0.022	0.024	0.022	0.002	8.3	0.338	0.342	1.1	0.67	0.62	7.4
0.4	0.041	0.045	0.042	0.003	6.6	0.32	0.336	4.7	1.42	1.3	8.4
0.6	0.06	0.065	0.072	0.007	9.7	0.314	0.33	4.8	2.19	2.01	8.2
0.8	0.079	0.085	0.086	0.001	1.2	0.309	0.327	5.5	2.97	2.72	8.4
1	0.097	0.104	0.12	0.016	13.3	0.306	0.321	4.6	3.75	3.45	8
1.2	0.115	0.124	0.128	0.004	3.1	0.303	0.317	4.4	4.53	4.18	7.7
1.4	0.133	0.143	0.15	0.007	4.7	0.302	0.311	2.8	5.32	4.91	7.7

Eq. (17) is a linearized form of Eq. (11) and the coefficient 'K_L' is a communicator of α and β. Therefore, Eq. (9) can be written as,

$$\frac{d^2\alpha}{dt^2} = \frac{-K_1K_2}{1+\frac{K_L}{K_{sh}}}\alpha^{\frac{3}{2}}. \tag{18}$$

After some manipulations we have,

$$\frac{1}{2}\left(\frac{d\alpha}{dt}\right)^2 - \frac{1}{2}\left(\frac{d\alpha_0}{dt}\right)^2 = -\frac{2}{5}\frac{K_1K_2}{1+\frac{K_L}{K_{sh}}}\alpha^{\frac{5}{2}}, \tag{19}$$

where α_0 is the approach at the initial condition which is equal to zero.

Furthermore, from Eq. (8) and initial condition combining with Eq. (17), we arrive at;

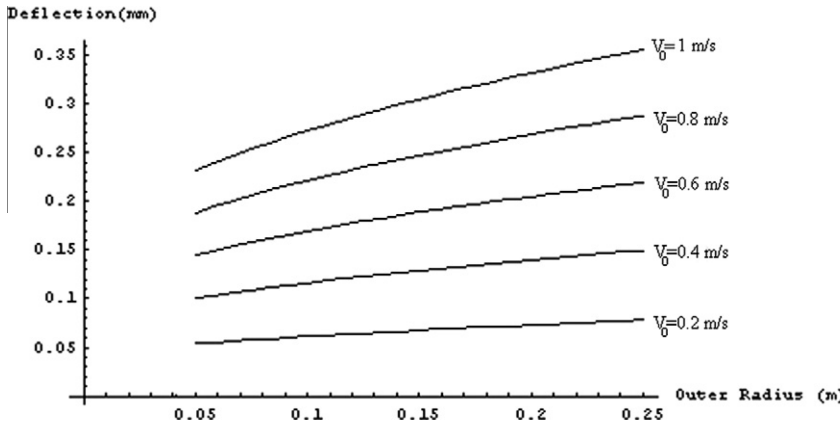

Fig. 2. Variations of shell deflection with respect to increase of outer radius when other parameters are kept constant (E_{sh} = 70 GPa, E_{sol} = 200 GPa, m_{sh} = 2 kg, Thickness = 5 mm).

$$V'_{sh} = \frac{d\alpha}{dt} + \frac{K_L}{K_{sh}}\frac{d\alpha}{dt} = \frac{d\alpha}{dt}\left(1 + \frac{K_L}{K_{sh}}\right), \tag{20}$$

$$\frac{d\alpha_0}{dt} = \frac{d\alpha}{dt}\bigg|_{\alpha=0} = \frac{V_0}{\left(1 + \frac{K_L}{K_{sh}}\right)}. \tag{21}$$

Thus, Eqs. (20) and (21) leads to;

$$\frac{d\alpha}{dt} = \sqrt{\frac{V_0^2}{\left(1 + \frac{K_L}{K_{sh}}\right)^2} - \frac{4}{5}\frac{K_1K_2}{1 + \frac{K_L}{K_{sh}}}\alpha^{\frac{5}{2}}}, \tag{22}$$

where $\frac{d\alpha}{dt}$: is the approach velocity.

At the moment where the maximum deflection of the spherical shell, α_{max}, with respect to the barrier occurs, the instant velocity of the center of the shell is equal to zero. Thus, equating Eq. (22) to zero, the maximum elastic deformation of the center of the shell (the approach) can be determined as,

$$\alpha_{max} = \left(\frac{5}{4}\frac{V_0^2}{\left(1 + \frac{V_0^2}{K_1K_{sh}}\left(\frac{4}{5}\frac{K_1K_2}{V_0^2}\right)^{\frac{4}{5}}\right)K_1K_2}\right)^{\frac{2}{5}}. \tag{23}$$

Also, the maximum membrane and bending deformation of the shell, β_{max} is determined using Eqs. (11) and (23), thus,

$$\beta_{max} = \frac{K_2}{K_{sh}}\left(\frac{5}{4}\frac{V_0^2}{\left(1 + \frac{V_0^2}{K_1K_{sh}}\left(\frac{4}{5}\frac{K_1K_2}{V_0^2}\right)^{\frac{4}{5}}\right)K_1K_2}\right)^{\frac{3}{5}}. \tag{24}$$

Therefore, the maximum total deflection can be determined by Eq. (1), then,

$$\Delta x_{max} = \left(\frac{5}{4}\frac{V_0^2}{\left(1 + \frac{V_0^2}{K_1K_{sh}}\left(\frac{4}{5}\frac{K_1K_2}{V_0^2}\right)^{\frac{4}{5}}\right)K_1K_2}\right)^{\frac{2}{5}} + \frac{K_2}{K_{sh}}\left(\frac{5}{4}\frac{V_0^2}{\left(1 + \frac{V_0^2}{K_1K_{sh}}\left(\frac{4}{5}\frac{K_1K_2}{V_0^2}\right)^{\frac{4}{5}}\right)K_1K_2}\right)^{\frac{3}{5}}. \tag{25}$$

The duration of the impact, which is an important parameter in the analysis of the impact, is determined by integrating the instantaneous velocity of the center of the shell during the contact, i.e., $V_\alpha = \frac{d\alpha}{dt}$. Therefore, after some manipulations, which lead to gamma function, the integration of Eq. (22) leads to

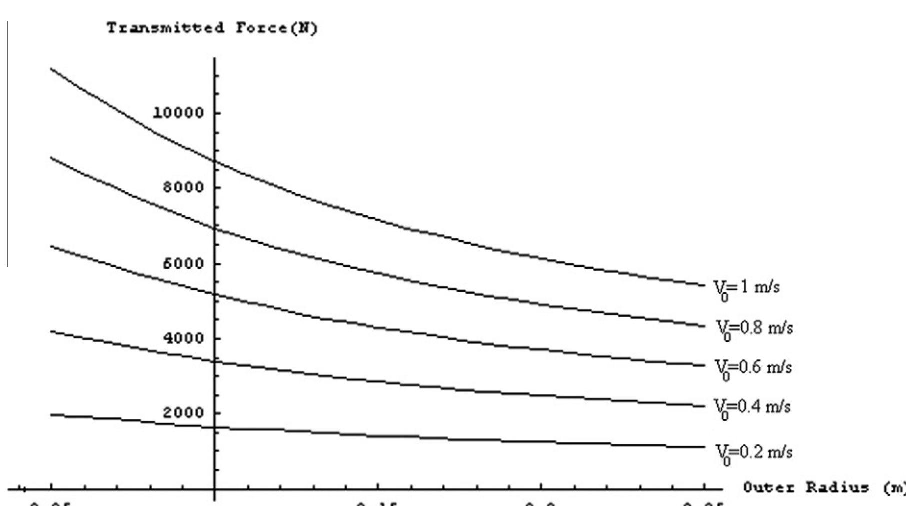

Fig. 3. Variations of maximum transmitted force with respect to increase of outer radius when other parameters are kept constant (E_{sh} = 70 GPa, E_{sol} = 200 GPa, m_{sh} = 2 kg, Thickness = 5 mm).

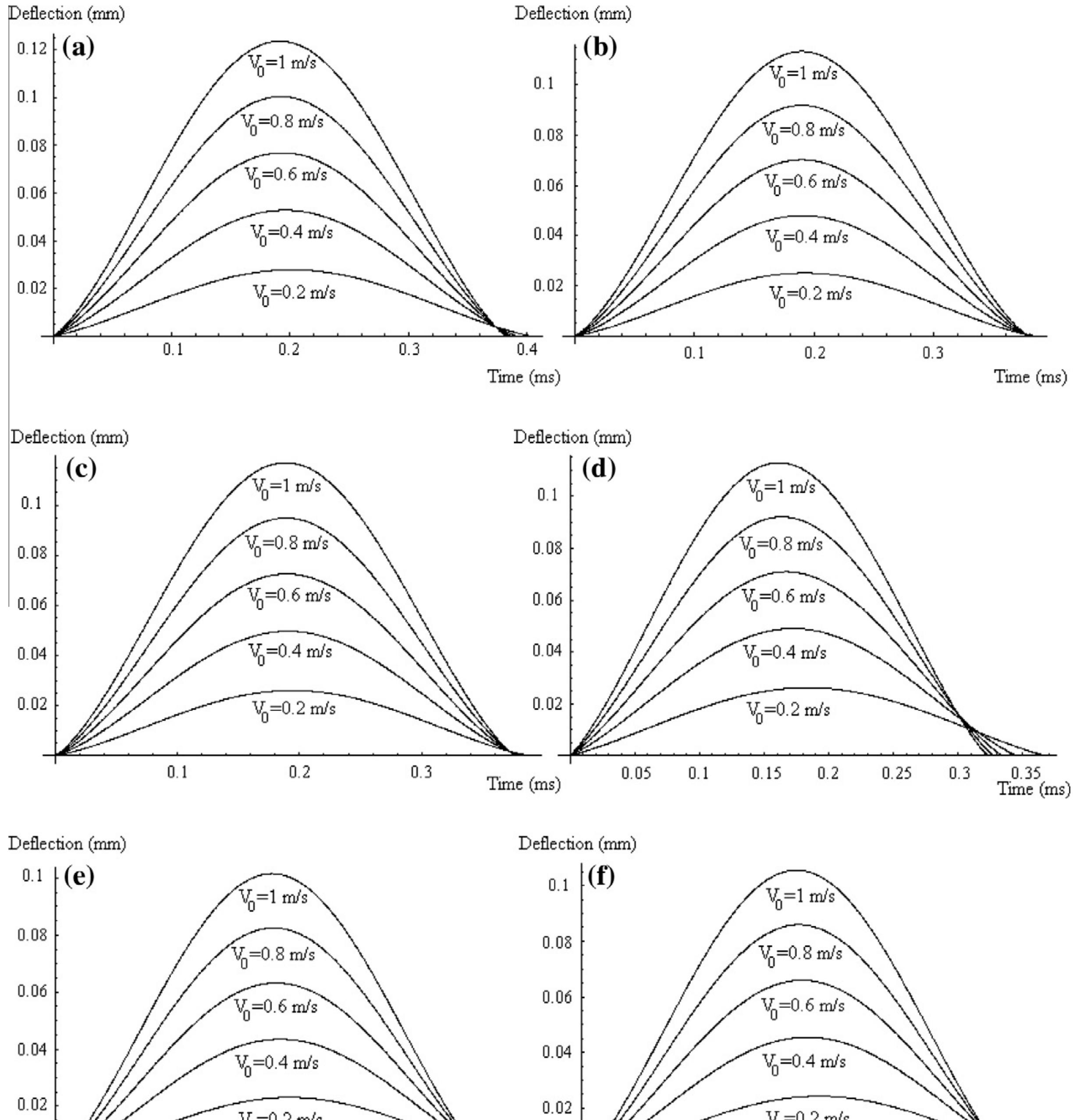

Fig. 4. Elastic deformation of spherical shell in different conditions listed below; (a) $\delta = 0.05$; $\rho_{sh} = 7800\ \mathrm{kg/m^3}$; $E_{sh} = E_{sol} = 200$ GPa. (b) $\delta = 0.05$; $\rho_{sh} = 7800\ \mathrm{kg/m^3}$; $E_{sh} = 200$ GPa; $E_{sol} \to \infty$. (c) $\delta = 0.05$; $\rho_{sh} = 2700\ \mathrm{kg/m^3}$; $E_{sh} = 70$ GPa; $E_{sol} = 200$ GPa. (d) $\delta = 0.08$; $\rho_{sh} = 7800\ \mathrm{kg/m^3}$; $E_{sh} = E_{sol} = 200$ GPa. (e) $\delta = 0.08$; $\rho_{sh} = 7800\ \mathrm{kg/m^3}$; $E_{sh} = 200$ GPa; $E_{sol} \to \infty$. (f) $\delta = 0.08$; $\rho_{sh} = 2700\ \mathrm{kg/m^3}$; $E_{sh} = 70$ GPa; $E_{sol} = 200$ GPa.

$$t_f = \frac{2\left(1+\frac{K_L}{K_{sh}}\right)}{V_0}\int_0^{\alpha_{\max}} \frac{d\alpha}{\sqrt{1-\frac{4}{5}\frac{K_1K_2}{V_0^2}\left(1+\frac{K_L}{K_{sh}}\right)\alpha^{\frac{5}{2}}}} = 2.94\left(1+\frac{K_L}{K_{sh}}\right)\frac{\alpha_{\max}}{V_0}, \tag{26}$$

where, t_f is the impact duration. Since the impact of the spherical shell to the flat barrier is assumed to be elastic and there is no energy dissipation in this process, thus the curve of deformation versus time is symmetric with respect to the relevant time of the maximum deflection. That is, the time for the compression is equal to the time of restitution of the spherical shell. Therefore, the total separation time t_f is twice the period of compression. That is why a factor 2 is placed in the first term of Eq. (26) before the integral.

Substituting for K_L from Eq. (16) leads to:

$$t_f = 2.94\left(1+\frac{V_0^2}{K_1K_{sh}}\left(\frac{4}{5}\frac{K_1K_2}{V_0^2}\right)^{\frac{4}{5}}\right)\frac{\alpha_{\max}}{V_0}. \tag{27}$$

Finally the maximum transmitted force during the impact is determined by Eq. (2), therefore,

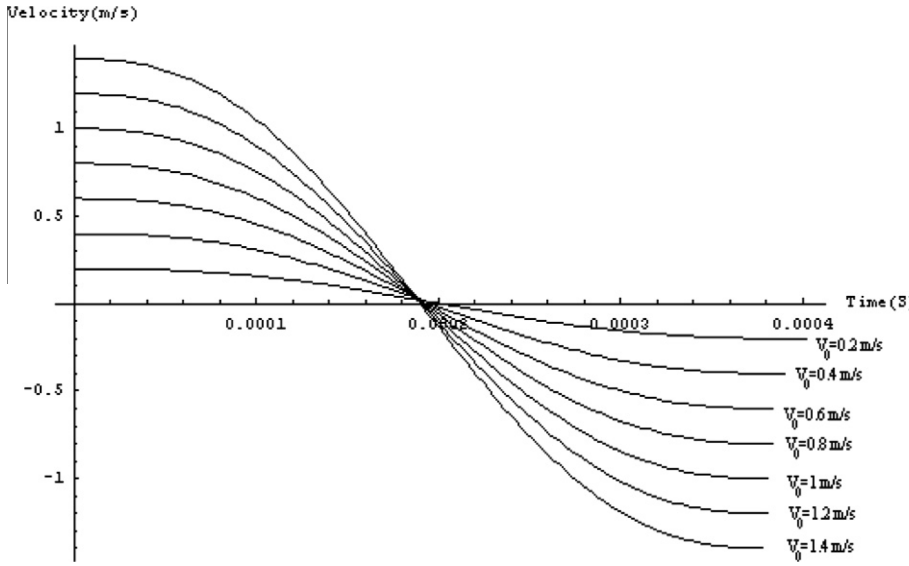

Fig. 5. The center velocity of the spherical shell during the impact for the values of δ = 0.05; ρ_{sh} = 7800 kg/m^3; E_{sh} = E_{sol} = 200 GPa.

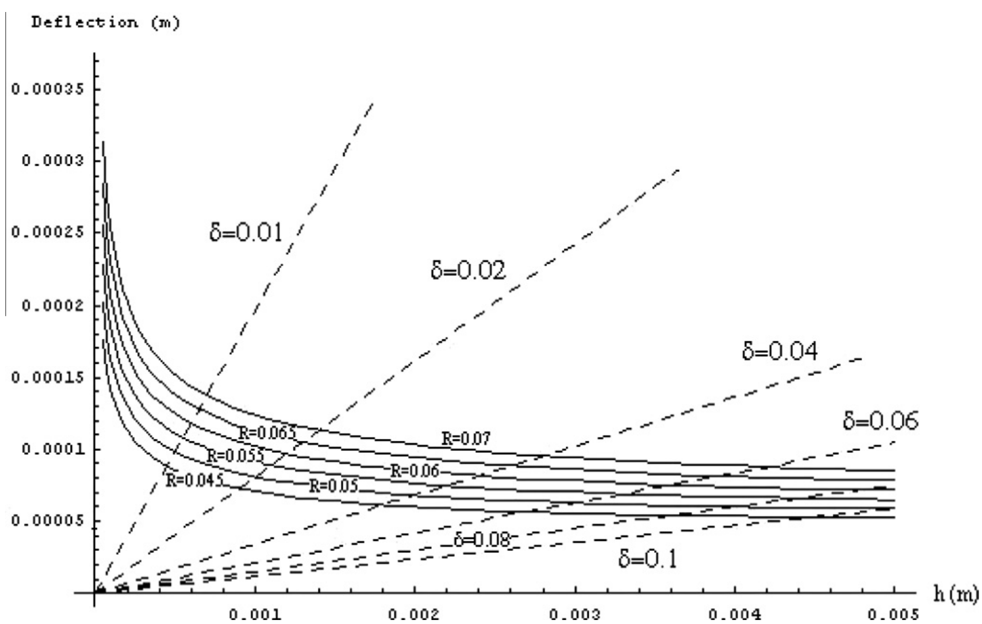

Fig. 6. Deformation dependency with respect to thickness obtained by Eq. (25) for the cases of V_0 = 0.5 m/s; ρ_{sh} = 7800 kg/m^3; E_{sh} = E_{sol} = 200 GPa.

$$F_{\max} = K_2\left(\frac{5}{4}\frac{V_0^2}{\left(1+\frac{V_0^2}{K_1K_{sh}}\left(\frac{4}{5}\frac{K_1K_2}{V_0^2}\right)^{\frac{4}{5}}\right)K_1K_2}\right)^{\frac{3}{5}}. \tag{28}$$

By applying the above method, the impact characteristics of the elastic shell were determined to analytically study the problem and its effective parameters.

The upper bound of the impact speed for spherical shell is needed in order to specify the accuracy range of the explicit expressions listed above. For this purpose the condition $\Delta x/R_{sh} \ll 1$ is applied and the upper bound velocity is determined. It is concluded that $\Delta x/R_{sh} \approx 0.004$ is a reasonable number where the conditions are satisfied. Therefore, the upper bound of the initial speed is V_{Upper} can be found from Eq. (29).

$$\left(\frac{5}{4}\frac{V_{Upper}^2}{\left(1+\frac{V_{Upper}^2}{K_1K_{sh}}\left(\frac{4}{5}\frac{K_1K_2}{V_{Upper}^2}\right)^{\frac{4}{5}}\right)K_1K_2}\right)^{\frac{2}{5}} + \frac{K_2}{K_{sh}}\left(\frac{5}{4}\frac{V_{Upper}^2}{\left(1+\frac{V_{Upper}^2}{K_1K_{sh}}\left(\frac{4}{5}\frac{K_1K_2}{V_{Upper}^2}\right)^{\frac{4}{5}}\right)K_1K_2}\right)^{\frac{3}{5}} = 0.004R_{sh}. \tag{29}$$

This equation can be numerically solved for specifying the accuracy range. This quantity is calculated for different thickness ratios and different materials and is given in Tables 1–6.

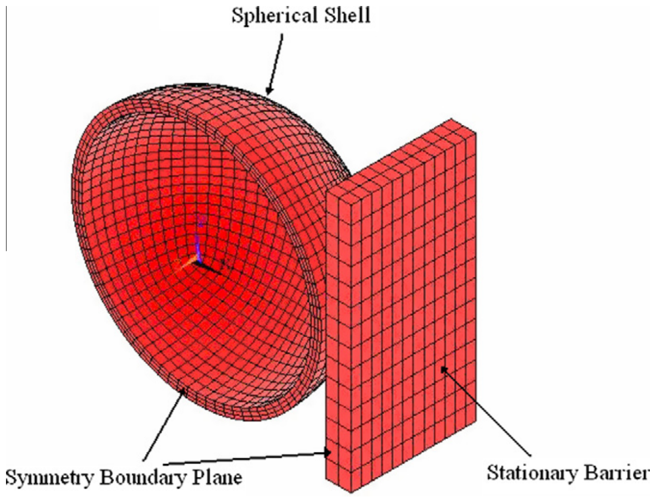

Fig. 7. The FE model of the problem.

3. Results

Utilizing the closed form solution the variables of the problem can be parametrically investigated. For example, using Eqs. (25) and (28) the maximum deflection of the shell and the maximum transmitted force as a function of outer radius are depicted in Figs. 2 and 3, respectively. The elastic total deformation of the spherical shell was determined by solving Eq. (18) which is a nonlinear differential equation and utilizing Eq. (11). Eq. (18) was solved numerically by developing a program based on Runge Kutta Fehlberg method (Gerald, 2003). Therefore, for different values of $\delta = h/R_{sh}$, the results are presented in Fig. 4. The center velocity of the spherical shell during the impact for the value of $\delta = 0.05$ is given in Fig. 5. On the other hand, the deformation dependency with respect to thickness obtained by Eq. (25) for different cases is shown in Fig. 6.

It can be deduced that the transmitted force during contact is decreased by increasing the outer radius, when other parameters including the total mass of the shell is constant, as shown in Fig. 3. On the other hand observation of Fig. 4 reveals that all of the curves are symmetric, since the impact condition between the shell and the half space is elastic and no energy dissipations take place. In addition, the time duration is decreased by the increase of the initial velocity and, finally the time duration is decreased by the increase of the thickness when other parameters are kept constant.

4. Model validation

Results of the stated closed form solutions of this problem are compared with Young's research work (Young, 2003) and also the results of the analysis using the finite element analysis.

Young (2003) investigated the impact of an elastic spherical shell based on conservation of mechanical energy and linear momentum. He obtained an implicit expression for the transmitted force of elastic shell at the maximum deflection, which is

$$m^*\Delta V^2 = \frac{F^2}{K_{sh}} + \frac{4}{5}\frac{F^{\frac{5}{3}}}{K_2^{\frac{2}{3}}}, \tag{30}$$

where

$$\frac{1}{m^*} = \frac{1}{m_{sh}} + \frac{1}{m_{sol}} \tag{31}$$

and

$$\Delta V = V_{sh} - V_{sol}. \tag{32}$$

Note that Eq. (30) cannot be solved analytically, thus, the explicit expression of the transmitted force could not be obtained and consequently, parametric study and analytical investigation with this equation is not achievable.

Young solved Eq. (30) numerically in order to obtain the transmitted force and consequently the elastic deformation due to contact (the approach), and the membrane and bending deformation of the elastic shell. Young's results were compared with the result of Eq. (28) and are presented in Tables 1–6.

As another means of validation for the closed form solution and the proposed model explicit finite element method was utilized. ANSYS/LS-DYNA Code (Ansys version , 2005) was used and the impact of a spherical shell with an elastic barrier was analyzed. The shell and the barrier were discretized and appropriate mesh for the bodies was generated. Solid elements type 164 with full integrated method was used. Appropriate contact characteristics for the surfaces of the objects were created. Due to the symmetry of the geometry only half of the bodies were modeled. The barrier was restrained and the spherical shell was subjected to an initial horizontal velocity condition. Fig. 7 illustrates the finite element model of the problem.

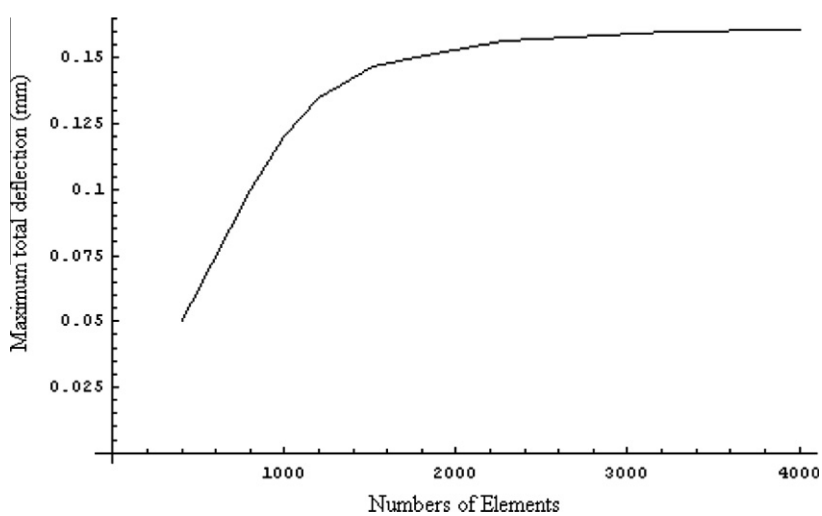

Fig. 8. Sensitivity analysis for the case of $\delta = 0.05$; $E_{sol} = E_{sh} = 200$ GPa; $V = 1.4$ m/s.

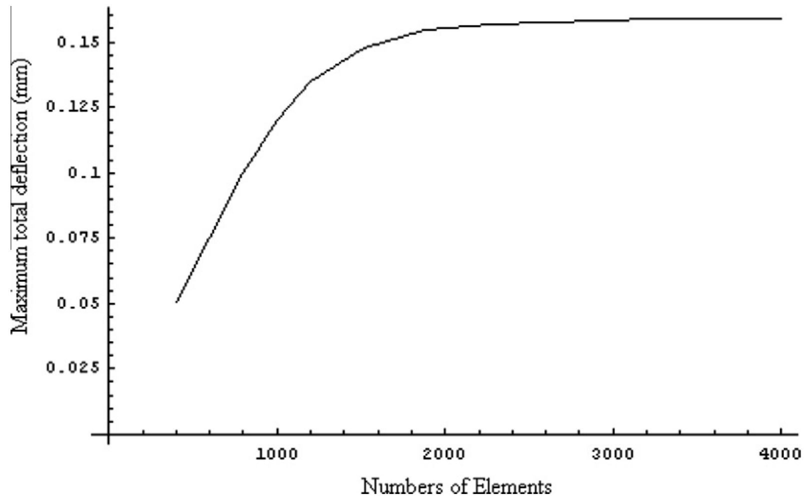

Fig. 9. Sensitivity analysis for the case of $\delta = 0.08$; $E_{sol} = E_{sh} = 200$ GPa; $V = 1.4$ m/s.

The spherical shell had a radius of 0.05 m and thicknesses of 0.0025 m and 0.004 m, and the stationary barrier had dimensions of 0.1 m $\times$ 0.05 m $\times$ 0.005 m. It is necessary to note that sensitivity analysis of the model was performed in order to determine the appropriate number of elements of the problem. That is, to minimize the error in a FE analysis, adaptive finite element method consists of element size refinement, higher order polynomial interpolations, combination of the element size and order of the polynomial and adjusting nodal position is utilized (Stein et al., 2007). In this paper, however, due to the simple geometry of the problem only the element size adaptive finite element method was utilized. That is, uniform refinement of the elements was employed to obtain the best precision of numerical method. As examples, mesh sensitivity of the finite element analysis for two cases ($\delta = 0.05$, $\delta = 0.08$) are shown in Figs. 8 and 9. Based on Figs. 8 and 9 it was determined that 3936 elements exceeded the saturation limit of numerical response.

Finally the impact of the spherical shell with the barrier was analyzed and numerical results were obtained. The maximum total deflections are determined by Eq. (25) and are compared with Young's results and the responses of the finite element model. Tables 1–6 reveal the comparison of the results of the three methods, the finite element analysis, Young's work and the analytical method, for different velocities and the shell thicknesses.

Tables 1–6 reveal the comparison of the results of the three methods, i.e., the element size adaptive finite element method, Young's results (Young, 2003), and the analytical method, for different velocities and the shell thicknesses.

The tables reflect a good agreement between the results of the closed form solutions, the finite element as well as the responses of Young's equation. It is observed that the percentages of difference in all cases are negligible, therefore it is concluded that the proposed closed form solution is valid for predicting the response of elastic spherical shell to low velocity impact.

5. Conclusions

In this paper a closed form solution for the problem involving the impact of a spherical hollow shell with a fixed elastic barrier was obtained. The solution was accomplished by implementing some simplifications of the equation. The reason for this approach is that the impact problem is highly nonlinear and does not have an analytical solution. A linearization of the equation was proposed and a closed form solution of the problem was obtained. In this study the coefficient of restitution was equal to one since the energy losses associated with friction, vibrations of the shell and elastic–plastic deformations were neglected.

The results of the closed form solution were compared with the work of Young (2003). Young investigated the impact between two elastic bodies and he employed Hertzian contact theory and conservation of mechanical energy and linear momentum at the maximum compression of two striking objects. Based on these theories, he proposed Eq. (30) where the transmitted force (F) is unknown. However, Eq. (30) cannot be solved analytically and numerical methods should be employed. That is, the final response of the Young's equation does not involve quantities that are effective to the impact force or other important characteristics, and thus, it is not suitable for parametric study of the problem. However, in the method presented in this paper, the Hertzian and Reissner (membrane–bending deformation theory) theories were employed for the deformation Eq. (1). In addition, in order to obtain an analytical solution to the problem, a linearized Hertzian contact theory based on deformation energy (potential energy) was employed. The linearization enabled us to obtain the closed form solution that involves important characteristics of the problem. The closed form solution facilitates a parametric study of the problem, i.e., the effect of each parameter on the impact force. Specifically, the closed form solution for the impact force, total deformation, and the time duration were obtained and can be studied parametrically. The closed form solution can explore the parametric relationship between output quantities (for example, impact force, duration of the impact, etc.) and initial quantities (for example, mass, geometry, velocity, etc.). Consequently analytical solutions are powerful methods in order to find the mathematical relationship between input quantities and output characteristics of the system.

In addition, the finite element method was utilized to validate the results of the closed form solution. To minimize the error in the FE analysis, the element size adaptive finite element method was utilized and optimum number of elements was determined. The mesh sensitivity of the finite element analysis for two cases ($\delta = 0.05$, $\delta = 0.08$) are shown in Figs. 8 and 9. It was determined that 3936 elements exceeded the saturation limit of numerical response. The results of the three methods (closed form,Young's and FE) were compared and revealed negligible difference between the proposed closed form solution of this study, the finite element solution and the Young's results (Young, 2003).

Utilizing the explicit expressions of this study, one can perform a parametric study of the impact of a shell structure with a barrier. It is necessary to note that the combined accuracy of Hertzian contact theory and Reissner effect strongly depends on the assumption of thin walled spherical shell and small deformation via neglecting the stress wave propagation. Therefore, a uniform acceleration of mass during the contact is considered.

This method can be applied for structures other than spherical shells with two degrees of curvature of the surface, if the Hertz theory can be employed for that problem. It implies that the initial contact of two isotropic elastic bodies should be a point not a surface and deformation would be small in comparison with other dimensions of each colliding object. Also if that body has membrane and bending deformation, there should be a mathematical relationship between the applied force and the membrane deformation for that particular structure. In addition, this analysis can be employed for analytical study of elastic deformation of human skull in low speed impact. Finally, the proposed closed form solution is appropriate for impact investigation and parametric evaluation of an elastic spherical shell.

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